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THE FORMULATION OF PROBLEMS OF LUBRICATION THEORY FOR JOURNAL BEARINGS†

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The formulation of problems of the flow in the clearance of a closed hydrodynamic journal bearing in the case of a "cylindrical" clearance is considered in the approximation of an incompressible or slightly compressible lubricant. The cylindrical shape of the clearance is determined by the fact that the fixed external surface shaped formed by the bearing is independent of the coordinate measured along the axis of the revolving shaft; hence the clearance is only a function of the angle variable. Bearings of a large (in the limit, infinite) relative length in the axial direction are of particular interest. The condition which, in the one-dimensional approximation, takes into account effects related to the outflow of lubricant from the open ends of the sections with (or without) a supply aperture is obtained. In special cases the well-known "supply aperture condition" and an integral equality, often called "the Elrod–Burgdorfer condition", which are used to consider bearings of infinite length, follow from this condition. The problems considered are particularly interesting when solving variational problems of the optimization of hydrodynamic bearings in the one-dimensional approximation.

1. THE EQUATIONS AND CONDITIONS FOR A BEARING OF FINITE LENGTH

Suppose the clearance h° is a function of only the angle variable θ in fixed cylindrical coordinates z°, r°, θ, with the axial variable z° measured from the plane of symmetry of the journal bearing (Fig. 1), i.e. $h^\circ = h^\circ(\theta)$, the shaft rotates with angular velocity ω° in an anticlockwise direction, R° is its radius and $2B^\circ$ is the length of the bearing. The superscript $^\circ$ denotes dimensional quantities. If $x^\circ = \theta R^\circ$ is the coordinate, measured from the plane $\theta = 0$ along the surface of the shaft in the direction of its rotation, p° is the pressure, T° is the temperature, $\rho^\circ = \rho^\circ(p^\circ, T^\circ)$ is the density and $\mu^\circ = \omega^\circ(T^\circ)$ is the viscosity of the lubricant, then, in the approximation of lubrication theory, the equation which determines the pressure distribution $p^\circ = p^\circ(x^\circ, z^\circ)$ in the lubrication layer when $T^\circ = T^\circ_* = \text{const}$ (the Reynolds equation) has the form [1]

$$(\rho h^3 p_x)_x + (\rho h^3 p_z)_z - (\rho h)_x = 0 \quad (\varphi_x = \partial\varphi/\partial x,\ \varphi_z = \partial\varphi/\partial z) \tag{1.1}$$

To obtain this equation the following dimensionless variables and the parameter γ are introduced.

$$x = \frac{x^\circ}{R^\circ},\ z = \frac{z^\circ}{R^\circ},\ B = \frac{B^\circ}{R^\circ},\ h = \frac{h^\circ}{h^\circ_*},\ \rho = \frac{\rho^\circ}{\rho^\circ_*},\ p = \frac{p^\circ}{\gamma\rho^\circ_* U^{\circ 2}},\ \mu = \frac{\mu^\circ}{\mu^\circ_*} = 1;\ \gamma = \frac{6R^\circ\mu^\circ_*}{\rho^\circ_* h^{\circ 2}_* U^\circ}$$

where h°_*, ρ°_* and $\mu^\circ_* = \mu^\circ(T^\circ_*)$ are the "characteristic" clearance, and the density and viscosity of the lubricant, respectively, taken as scales.

For an incompressible lubricant $\rho = 1$, and for a slightly compressible lubricant $\rho = 1 + (p - p_*)\gamma U^{\circ 2}/a^{\circ 2}_*$ for the corresponding choice of ρ°_* and p°_*, where $1/a^{\circ 2}_* = (\partial\rho^\circ/\partial p^\circ)_{T^\circ}$ when $p^\circ = p^\circ_*$ and $T^\circ = T^\circ_*$. Since a°_* is the isothermal velocity of sound, $U^{\circ 2}/a^{\circ 2}_* = M^2$ is the square of the isothermal Mach number, determined by a°_* and by the velocity of motion U° of the shaft surface. If the magnitude of M^2 is so small that $(p - p_*)\gamma M^2 \ll 1$, then $\rho \approx 1$ and Eq. (1.1) takes a form identical to the equation for an incompressible lubricant

$$(h^3 p_x)_x + (h^3 p_z)_z - h_x = 0 \tag{1.2}$$

Henceforth we will draw no distinction between incompressible and slightly compressible lubricants.

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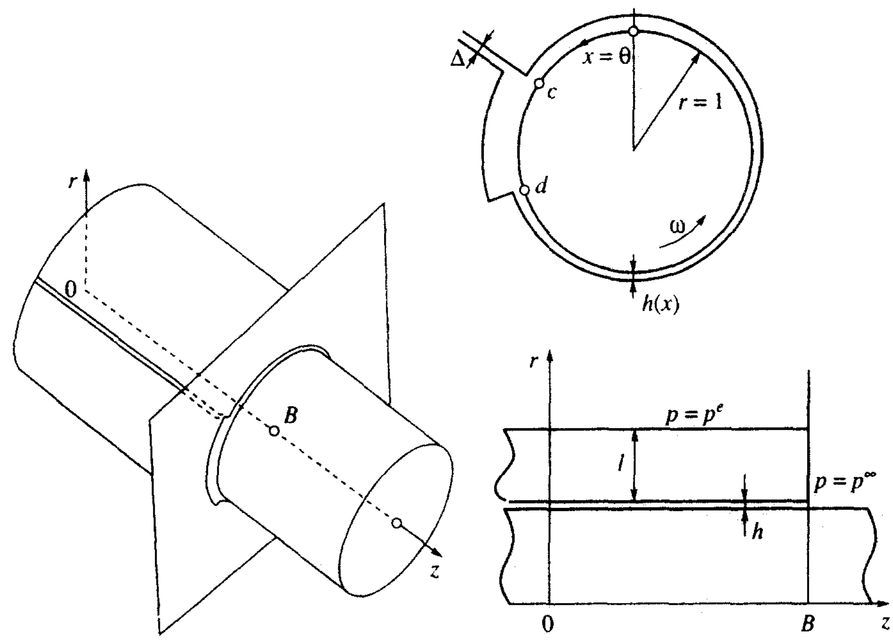

Fig. 1

We will write the boundary conditions which the dimensionless pressure distribution $p(x, z)$ must satisfy together with Eq. (1.2) for a specified periodic function $h(x)$. We will start with the conditions for p and the flow rate to be periodic and continuous in the sections $x = x_d$ of a possible sudden change in the clearance

$$p(L,z)=p(0,z),\quad p_x(L,z)=p_x(0,z),\quad -B\leqslant z\leqslant B \tag{1.3}$$

$$[p(x_d,z)]=0,\quad [h_d^3 p_x(x_d,z)-h_d]=0,\quad -B\leqslant z\leqslant B \tag{1.4}$$

Here $L = 2\pi$, $[\varphi] = \varphi_+ - \varphi_-$, and the $-$ and $+$ subscripts denote the values of parameters before and after the discontinuity in the direction in which θ or x increase.

If the radial "supply aperture" situated at $x = x_c$ is parallel to the axis of rotation, this section of the clearance will be considered as the surface of discontinuity of the discharge under continuous pressure. We will use the same approximation of lubrication theory to determine the flow entering the clearance or leaving it through the aperture. Suppose the width of the aperture Δ° be much smaller than its length in the radial direction $l^\circ = O(R^\circ) \ll B^\circ$. Neglecting the flow in the aperture in the direction of the z axis in view of the last inequality, we find that double the flow of lubricant through the aperture equals $q^\circ = q^\circ(z) = \Delta^{\circ 3}\rho^\circ(p^{e\circ} - p_c^\circ)/l^\circ$, where $p^{e\circ}$ is the external pressure above the aperture and p_c° is the pressure in the clearance when $x = x_c$. After changing to dimensionless quantities, we obtain $q = K(p^e - p_c)$, where $K = \Delta^3/l$, $l = l^\circ/R^\circ$ and $\Delta = \Delta^\circ/h_*^\circ$, and consequently

$$[p(x_c,z)]=0,\quad [h_c^3 p_x(x_c,z)-h_c]+K(p^e-p_c)=0,\quad -B\leqslant z\leqslant B \tag{1.5}$$

This condition holds in this notation when $x_c = x_d$ i.e. when the coordinates of the aperture and the abrupt change in the clearance are identical. Using the arbitrariness of the choice of the origin of the circumferential coordinate, we will henceforth assume that the coordinates x_c and x_d differ from 0 and 2π.

In the approximation of lubrication theory the pressure at the open ends of the bearing (where $z = \pm B$) equals the ambient pressure p^∞ in the medium around the shaft outside the bearing. In the general case (when there are special isolating diaphragms) it differs from the "external" pressure p^e above the aperture. Putting $p^\infty = 0$, i.e. measuring the pressure p with respect to p^∞ we obtain

$$p(x,B)=p(x,-B)=0,\quad 0\leqslant x\leqslant L \tag{1.6}$$

Taking into account the symmetry of the flow in the bearing with respect to the plane $z = 0$, it is more convenient to consider half of it: $0 \leqslant z \leqslant B$. In this case the symmetry condition

$$p_z(x, 0) = 0, \quad 0 < x \leqslant L \tag{1.7}$$

is used instead of the equality $p(x, -B) = 0$.

2. THE SOLUTION OF THE PROBLEM AND ITS PROPERTIES IN THE CASE OF LONG BEARINGS

To investigate the properties of the solution of problem (1.2)–(1.7) required later, we will first represent the pressure distribution as a sum

$$p(x, z) = p_0(x) + p_1(x, z) \tag{2.1}$$

with the function $p_0(x)$ satisfying the equation

$$(h^3 p_{0x})_x - h_x = 0 \tag{2.2}$$

and conditions (1.3)–(1.5) with p and p_0 replaced, and the partial derivatives with respect to x replaced by total derivatives. The solution $p_0(x)$ of this one-dimensional problem, with the given piecewise-continuous periodic function $h(x)$ exists and is unique for a supply aperture, and when there is no aperture the solution is determined apart from an arbitrary additive constant. The uniqueness of the solution when there is an aperture results from the conditions for $p_0(x)$ obtained from (1.3)–(1.5), which lead to the equality $p_0(x_c) = p^e$.

Taking expression (2.1), Eq. (2.2) and the above-mentioned properties of the solution $p_0(x)$ into account, as well as relations (1.2)–(1.7), the problem of determining $p_1(x, z)$ takes the form ($f = h^3$)

$$(fp_{1x})_x + (fp_{1z})_z = 0 \tag{2.3}$$

$$p_1(L,z) = p_1(0,z), \; p_{1x}(L,z) = p_{1x}(0,z), \; [p_1(x_d,z)] = 0, \; [f_d p_{1x}(x_d,z)] = 0$$
$$[p_1(x_c,z)] = 0, \; [f_c p_{1x}(x_c,z)] - Kp_{1c} = 0, \quad 0 \leqslant z \leqslant B \tag{2.4}$$

$$p_1(x,B) = -p_0(x), \; p_{1z}(x,0) = 0, \quad 0 \leqslant x \leqslant L \tag{2.5}$$

Unlike the initial boundary-value problem for $p(x, z)$, boundary-value problem (2.3)–(2.5) for $p_1(x, z)$ allows of a separation of variables. Doing this and substituting $p_1(x, z) = X(x)Z(z)$ into (2.3)–(2.5), we obtain the following equations for determining the functions X and Z (the prime denotes a total derivative with respect to x and a dot above a symbol denotes a total derivative with respect to z)

$$(fX')' + \lambda fX = 0 \tag{2.6}$$

$$\ddot{Z} - \lambda Z = 0 \tag{2.7}$$

and the conditions

$$X(L) = X(0), \quad X'(L) = X'(0), \; [X]_d = 0, \; [fX']_d = 0, \; [X]_c = 0, \; [fX']_c - KX_c = 0 \tag{2.8}$$

$$\dot{Z}(0) = 0 \tag{2.9}$$

Here λ is the eigenvalue (EV) of homogeneous boundary-value problem (2.6), (2.8). We will consider the functional (everywhere henceforth integration with respect to x is carried out from 0 to L)

$$I(X) = KX_c^2 + \int \{f(X')^2 - \lambda fX^2\} dx$$

defined for any bounded piecewise-continuous functions f, X and X', to determine the sign of all its EVs. We will write the first term of the integrand as the product $(fX')X'$ and integrate by parts over continuous sections of the functions f and X', taking conditions (2.8) and the periodicity of the function

f into account. For any eigenfunction (EF) X, which is the solution of boundary-value problem (2.6), (2.8), we then obtain that $I(X) = 0$. By definition $f = h^3 > 0$, hence for $K > 0$ the equality $(X) = 0$ is only possible for positive λ, i.e. when

$$\lambda_k = \mu_k^2 > 0, \quad k = 1, 2, \ldots \tag{2.10}$$

If there is no supply aperture ($K = 0$), the minimum EV $\lambda_0 = \mu_0^2 = 0$. Its corresponding EF X_0 is a constant. In addition, according to (2.7) and (2.9), $Z_0 = 1$, the function $p_0(x)$ is defined, apart from an additive constant, and condition (2.10) holds for $k \geqslant 1$, as well as for $K > 0$.

By virtue of relations (2.9) and (2.10), the solutions $Z_k = Z_k(z)$ of Eq. (2.8) have the form

$$Z_k = \operatorname{ch} \mu_k z \tag{2.11}$$

Let $X_k = X_k(x)$ be the EF, corresponding to the EV λ_k. The EV of problem (2.6), (2.8), which is also self-adjoint (with natural supplementations for $x = x_d$ and $x = x_c$), forms a series which increases without limit, as in the standard Sturm–Liuville self-adjoint problem [2], and the corresponding EFs X_k are orthogonal and can be made orthonormal, i.e.

$$\int f X_m X_n dx = \delta_{mn} \tag{2.12}$$

where δ_{mn} is the Kronecker delta. In particular, by virtue of (2.12)

$$X_0 = \left(\int f dx\right)^{-1/2} \tag{2.13}$$

By conditions (2.8), the completeness of EF is proved in the same way as in [2] for the function f with a continuous first derivative. If the function f is continuous, it can be shown as in [2] that $\lambda_k \to k^2$ as $k \to \infty$ and the corresponding EFs become close to the superposition of $\sin kx$ and $\cos kx$. This property does not apply to discontinuous functions f and these "integer-valued" periodic functions do not approach the EF of problem (2.6), (2.8) even for piecewise-constant f. The neglect of this extremely important fact should not affect the results obtained in [3] for stepped bearings of finite length (apart from the fact that it is erroneously assumed in [3] that $\lambda_k = -k^2 < 0$).

Taking (2.11) into account we obtain

$$p_1(x, z) = p_{10} + \sum a_k X_k \operatorname{ch} \mu_k z \tag{2.14}$$

Here and below, the summation with respect to the subscript is carried out from $k = 1$ to $k = \infty$, $\mu_k > 0$ and $p_{10} \neq 0$ only when $K = 0$.

We substitute expression (2.14) into the first, unsatisfied condition of (2.5) to determine the constant p_{10} and the coefficients a_k, and by carrying out the well-known procedure, based on (2.12) and (2.13), we obtain

$$p_{10} = -\left(\int f(x) dx\right)^{-1} \int f(x) p_0(x) dx, \quad K = 0$$

$$a_k = \frac{\alpha_k}{\operatorname{ch} \mu_k B}, \quad \alpha_k = -\int f(x) p_0(x) X_k(x) dx, \quad k = 1, 2, \ldots \tag{2.15}$$

For a bearing without a supply aperture, the condition $B \gg 1$ was not used when obtaining solution (2.14) with p_{10} and a_k, given by (2.15). When there is an aperture, less restrictive, for $l < 1$, inequality $B \gg l$ was used indirectly (at the model level it was used to justify the radial character of the flow in it).

Equations (2.14) and (2.15) contain the EVs and EFs, the determination of which is often a difficult problem. We will demonstrate this by two examples.

We will consider, as the first example, a piecewise-constant clearance with two steps and without a supply aperture. In the one-dimensional approximation, journal bearings of this type provide the maximum bearing capacity [4–8]. Suppose (Fig. 2a) $f = 1$ when $0 \leqslant x < \alpha$ and when $\beta < x \leqslant 2\pi$, and $f = F > 1$ when $\alpha < x < \beta$. In this case, according to Eq. (2.6) and the corresponding conditions from (2.8), the EFs are obtained as various combinations of $\sin \mu_k x$ and $\cos \mu_k x$ when $0 \leqslant x < \alpha$ and when $\alpha < x < \beta$, and $\sin \mu_k (x-2\pi)$ and $\cos \mu_k (x-2\pi)$ when $\beta < x \leqslant 2\pi$. In this case μ_k are the roots of the equation ($\mu_k = \pm\mu$)

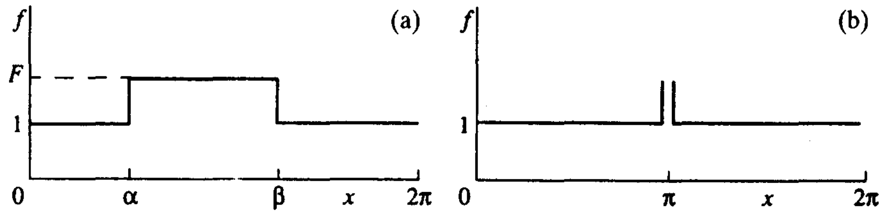

Fig. 2

$$a \cdot b - c \cdot d = 0$$

$$a(\mu) = -a(-\mu) = \cos\mu\alpha \sin\mu(\beta-\alpha) + F\{\sin\mu\alpha\cos\mu(\alpha-\beta) - \sin\mu(\beta-2\pi)\}$$

$$b(\mu) = b(-\mu) = \sin\mu(\alpha-\beta)\sin\mu(\beta-2\pi) + F\{\cos\mu\alpha - \cos\mu(\alpha-\beta)\cos\mu(\beta-2\pi)\}$$

$$c(\mu) = c(-\mu) = \sin\mu\alpha\sin\mu(\alpha-\beta) + F\{\cos\mu\alpha\cos\mu(\alpha+\beta) - \cos\mu(\beta-2\pi)\}$$

$$d(\mu) = -d(-\mu) = \sin\mu(\beta-\alpha)\cos\mu(\beta-2\pi) + F\{\sin\mu\alpha - \sin\mu(\beta-2\pi)\cos\mu(\alpha-\beta)\}$$

The finite number of the roots of this transcendental equation can only be found numerically. In the case of arbitrary $F > 1$ and $0 < \alpha < \beta < 2\pi$ its roots $\mu = \mu_k$ for any integral are not close to k.

In the second, simpler, example, for a fixed clearance ($f = 1$ for $0 \leqslant x \leqslant 2\pi$) the supply aperture (Fig. 2b) is situated in the section $x = x_c = \pi$. In this case $p_0(x) = p^e$, all EVs $\lambda_k = \mu_k^2 > 0$ and the EFs are equal to $X_k(x) = A_k \cos \mu_k x$ when $0 \leqslant x < \pi$ and to $X_k(x) = A_k \cos \mu_k(x - 2\pi)$ when $\pi < x \leqslant 2\pi$ with the normalizing factor A_k by virtue of the conditions of periodicity and continuity in the section $x = \pi$. The second condition for X_k when $x = \pi$ gives the equation ($\mu_k = \pm\mu$)

$$\sin\mu\pi = K(2\mu)^{-1}\cos\mu\pi$$

that determines the EVs, which, unlike the first example, will be of the order of k^2 for large k. The pressure distribution in the clearance is written in the form

$$p(x,z) = p^e + \sum \alpha_k X_k \frac{\operatorname{ch}\mu_k z}{\operatorname{ch}\mu_k B}, \quad \alpha_k = -p^e \int X_k(x)dx$$

taking into account the condition $p(x, B) = p^\infty = 0$.

3. EQUATIONS AND CONDITIONS FOR THE PRESSURE AVERAGED OVER THE LENGTH OF THE BEARING

We will use the results obtained to justify the formulation of the boundary conditions for long (in the limit, infinite) bearings, which are freely in contact with the "external space" at the ends.

If $B \gg 1$, we then obtain from relation (2.1), taking (2.14) and (2.15) into account for $z > 0$

$$p(x,z) = p_0(x) + p_{10} + \sum \alpha_k X_k \frac{\operatorname{ch}\mu_k z}{\operatorname{ch}\mu_k B} \approx p_0(x) + p_{10} + \sum \alpha_k X_k e^{\mu_k(z-B)} \tag{3.1}$$

due to the fact that the μ_k are positive.

We will introduce a pressure averaged over the length of the lubricating layer

$$\langle p \rangle = \frac{1}{2B}\int_{-B}^{B} p(x,z)dz = \frac{1}{B}\int_0^B p(x,z)dz \tag{3.2}$$

Like p_0, $\langle p \rangle$ is a function of x only. The projections of the vector of the carrying capacity N are determined by integrating $\langle p \rangle \sin x$ and $\langle p \rangle \cos x$ with respect to x from 0 to $L = 2\pi$. Assuming $p(x, z) = \langle p \rangle + \Delta p(x, z)$, according to (3.2) we obtain

$$\int_0^B \Delta p(x,z)dz = 0 \tag{3.3}$$

Averaging Eq. (1.2) over z and taking Eq. (3.3) into account, we obtain

$$\frac{d}{dx}\left(h^3\frac{d\langle p\rangle}{dx}\right)-\frac{dh}{dx}=-\frac{1}{B}h^3\left.\frac{\partial p}{\partial z}\right|_{z=B}=O\left(\frac{1}{B}\right) \tag{3.4}$$

Analogues of conditions (1.3)–(1.5) to determine $\langle p\rangle$ as a function of x from Eq. (3.4) with zero right-hand side are obtained by applying the operation of averaging (3.2) and Eq. (3.3) to relations (1.3)–(1.5). The resulting equations and conditions for $p_0(x)$ and $\langle p\rangle$ are identical. This would seem to lead to the identity of $p_0(x)$ and $\langle p\rangle$. However, when there is a supply aperture ($K > 0$) the uniqueness of the solution $\langle p\rangle$ would be ensured by the condition $\langle p\rangle_c = p^e$, similar to the same condition for $p_0(x)$. In fact, there is a fundamental difference between $p_0(x)$ and $\langle p\rangle$ for any arbitrary finite B. Since $p_0(x)$ is the solution of a one-dimensional problem, there is no flow of lubricant caused by $p_0(x)$ in the axial direction, since it is independent of z. Consequently the flow of lubricant, caused by $p_0(x)$, in the circumferential direction is constant. Its constancy, together with conditions (1.3)–(1.5) and with $p_0(x)$ substituted for $p(x, z)$ results in the equality $p_{0c} = p^e$. In contrast to this, $\langle p\rangle$ is the result of averaging $p(x, z)$, and hence, at least for finite B, the constancy of the flow of lubricant caused by $p_0(x)$ in the circumferential direction in no way follows, and the equality $\langle p\rangle_c = p^e$ does not follow from conditions (1.3)–(1.5) with $\langle p\rangle$ substituted for $p(x, z)$. The second example, which was considered at the end of the previous section, serves as an additional illustration of this.

To obtain the condition which replaces the equality mentioned, we will write the equation of the flow of lubricant from the plane of symmetry $z = 0$ in the direction of the axis of rotation. When there is supply aperture we obtain

$$\frac{dG^z}{dz}=K(p^e-p_c),\quad G^z=G^z(z)=-\int h^3(x)p_z(x,z)dx=-\frac{dI_1(z)}{dz} \tag{3.5}$$

$$I_1(z)=\int h^3(x)p(x,z)dx$$

Integrating the first equation of (3.5) with respect to z from $z = 0$ to an arbitrary $z \leqslant B$, when $G^z(0) = 0$, we get

$$\frac{dI_1(z)}{dz}-K\int_0^z\{p_c(\zeta)-p^e\}d\zeta=0$$

Integrating this equality with respect to z from an arbitrary positive $z < B$ to $z = B$, where, due to the choice of the reference level pressure measurements, $p(x, B) = p^\infty = 0$, we find

$$I_1(z)-K\int_z^B\int_0^\xi\{p^e-p_c(\zeta)\}d\zeta d\xi=0 \tag{3.6}$$

This equality is a consequence of the integral law of conservation of the lubricant flow (3.5), formulated taking into account the periodicity of the flow, the symmetry condition for $z = 0$ and the fact that the lubricant pressure is equal to the external (zero) pressure at the ends $z = B$. When there is no supply aperture, i.e. for $K = 0$, relation (3.6) is equivalent to the total radial lubricant flow (integrated over a period) through any section of the lubricant layer $0 \leqslant z = \text{const} \leqslant B$ being equal to zero.

We will average condition (3.6), taking into account (3.3) and using the expression for p_c obtained from (3.1) to integrate it. As a result we obtain

$$I_2-\frac{KB^2}{3}\left\{p^e-p_{0c}-p_{10}-\frac{1}{3B^2}\sum\frac{\alpha_k}{\mu_k^2}X_k(x_c)+o\left(\frac{1}{B^2}\right)\right\}=0,\quad I_2=\int h^3(x)\langle p\rangle dx$$

When $B \gg 1$, substitution of (3.1) into (3.2) gives $\langle p\rangle = p_0(x) + p_{10} + O(1/B)$, and for any $K \geqslant 0$ we will have

$$I_2-KB^2(p^e-\langle p\rangle_c)/3=0 \tag{3.7}$$

In the limiting cases, when KB^2 is much greater or much less than unity, this equality reduces to the well-known conditions, which are used for bearings of infinite length. Thus, when there is a supply aperture and for any K as small as desired but non-zero, for a journal bearing of infinite length it can be reduced to the equality

$$\langle p \rangle_c = p^e \tag{3.8}$$

The same condition is obtained from (3.7) for finite B if $KB^2 \gg 1$, despite the possible smallness of $K = \Delta^3/l$. On the contrary, for $K = 0$ or for $K > 0$ but $KB^2 \ll 1$ Eq. (3.7) is replaced by

$$I_2 = 0 \tag{3.9}$$

Summarizing all the cases considered for $B \gg 1$, we obtain the conditions

$$\begin{gathered}\langle p \rangle_c = p^e \text{ for } KB^2 \gg 1;\ I_2 = 0 \text{ for } KB^2 \ll 1 \\ I_2 - KB^2(p^e - \langle p \rangle_c)/3 = 0 \text{ for } KB^2 = O(1)\end{gathered} \tag{3.10}$$

Any of these conditions acts as an additional one, which, for $B \gg 1$, the pressure $\langle p \rangle$, determined from the solution of the above one-dimensional problem, should satisfy. Earlier [1], in the general case of a compressible lubricant, the first equality of (3.10) was postulated as one of the additional conditions for a closed journal bearing of infinite length. The second condition of (3.10) is identical with the result of the analogous condition obtained for the infinite journal bearing without a supply aperture in the case of a compressible barotropic, partly isothermal or polytropic lubricant [1, 9–11].

Note that the magnitude of p_{10} is given by the first relation of (2.15) for $K = 0$ so that Eq. (3.9), by substituting $\langle p \rangle = p_0(x) + p_{10}$ into it, transforms to an identity due to this definition. The latter is natural since p_{10} is determined as a coefficient of the expansion in the eigenfunction X_0 = const of the condition $p(x, B) = 0$. If the one-dimensional problem of finding $\langle p \rangle$ from Eq. (3.6), the conditions of periodicity, etc. is initially considered, then the level of measurement of $\langle p \rangle$, in no way related to $p^\infty = 0$ when solving this problem, for $K = 0$ will be determined by condition (3.9).

In the case of an incompressible lubricant, $\langle p \rangle$ is determined, apart from an additive constant, which, due to the multiplication of the pressure by $\sin x$ and $\cos x$ and the integration with respect to x from 0 to 2π for closed journal bearings, does not affect the magnitude and direction of the bearing capacity N, and thus in such cases each of conditions (3.10) can be replaced by specifying an arbitrary initial value of p for any x; for example, we can put $\langle p \rangle = 0$ for $x = 0$. Moreover, this arbitrariness enables any of the three conditions mentioned to be satisfied. For the same reason the inclusion of the second or third of these conditions as an isoperimetric condition to solve the previously considered variational problems [4–8] of the profiling of the clearance of a closed journal bearing of infinite length, cannot affect the shape of the optimum clearances for an incompressible lubricant. Finally, in view of the above considerations, these conditions are also optimal for closed cylindrical journal bearings of finite length but long enough ($B \gg 1$ and KB^2 are arbitrary).

When considering a slider, $x°$, $y°$ and $z°$ are fixed (connected to the slider) dimensional Cartesian coordinates, where the plane $y° = 0$ moves with velocity $U°$ in the direction of the $x°$ axis, and $y° = h°(x°)$ is the dimensional width of the clearance. If the clearance of such a slider is a periodic function $x°$ with period $2\pi R$, then all the equations and conditions obtained above will remain unchanged. In this case, however, the bearing capacity along the y axis, depends on the level of the pressure and hence on which of conditions (3.10) determines it.

4. THE CASE OF DIFFERENT PRESSURES AT THE ENDS

In the above discussion the pressure was assumed to be the same at both ends. We will consider the case when these pressures are different. Taking the half-sum of the pressure levels at the ends as the measurement reference level of the pressure p instead of condition (1.6) we will obtain

$$-p(x, -B) = p(x, B) = p^\infty \neq 0, \quad 0 \leqslant x \leqslant L \tag{4.1}$$

To fix our ideas we will assume that $p^\infty > 0$. In addition, if there is no supply aperture, the axial flow of lubricant in the clearance, which decreases as B increases, is negative. In this most simple case we will represent the solution in the form

$$p(x, z) = p^0(z) + p_0(x) + p_1(x, z) \tag{4.2}$$

with the function $p^0(z)$, which satisfies the equation $d^2p^0(z)/dz^2 = 0$ and boundary conditions (4.1). Taking into account that the integral of $p^0(z)$ in equality (3.3) for $\langle p\rangle$ equals zero, obtain that $p^0(z) = p^\infty z/B$, and for $p_0(x), p_1(x, z)$ and $\langle p\rangle$ practically all the above conclusion hold. The only difference is the natural correction of condition (3.9), which is replaced by

$$I_3 = 0,\ \ I_3 = \int h^3(x)(\langle p\rangle - p^\infty)dx \tag{4.3}$$

In the case of a clearance with a supply aperture, when the solution is represented in the form (4.2) the function $p^0(z)$ appears in the condition at the aperture, hindering the separation of the variables. Hence, for $p^0(z)$, preserving the splitting (2.1) of $p(x, z)$ into $p_0(x)$ and $p_1(x, z)$, and taking into account the oddness of boundary condition (4.1), we will replace conditions (2.5) by

$$p_1(x, B) = -p_0(x) + p^\infty,\ \ p_1(x, 0) = 0,\ \ 0 \leqslant x \leqslant L \tag{4.4}$$

Changing the condition for $z = 0$ leads to the replacement of the solution (2.11) by $Z_k = \operatorname{sh}\mu_k z$ for the previous determination of μ_k and X_k. Due to this conditions and (4.4), the solution for $p_1(x, z)$ becomes

$$p_1(x,z) = \sum \alpha_k X_k \frac{\operatorname{sh}\mu_k z}{\operatorname{sh}\mu_k B},\ \ \alpha_k = -\int f(x)\{p_0(x) + p^\infty\}X_k(x)dx$$

Hence, for $B \gg 1$, due to the fact that the μ_k are positive for $z > 0$ we obtain

$$p(x,z) = p_0(x) + \sum \alpha_k X_k \frac{\operatorname{sh}\mu_k z}{\operatorname{sh}\mu_k B} \approx p_0(x) + \sum \alpha_k X_k e^{\mu_k(z-B)} \tag{4.5}$$

Taking this into account we obtain

$$G^z(0) = \sum \frac{\alpha_k\mu_k}{\operatorname{sh}\mu_k B}\int f(x)X_k(x)dx \approx 2\sum \alpha_k\mu_k e^{-\mu_k B}\int f(x)X_k(x)dx$$

Using this equality and (4.5), we arrive at the conditions, which are obtained from (3.10) by replacing the integral I_2 by the integral I_3, defined by the second equality of (4.3). These conditions, for different pressure levels at the ends of the bearing, play the same role as conditions (3.10). We recall that p^∞ is the dimensionless pressure at the end $z = B$, which is measured from the half-sum of the pressure levels at the ends. If these pressure levels are equal, then $p^\infty = 0$ and the new conditions reduce to conditions (3.10).

If there are several supply apertures, we will denote the respective parameters by an additional subscript i, arranging the numbers $i = 1, \ldots, I \geqslant 2$ in order of increasing K_i. Then conditions (3.10) and the corresponding condition for different pressure levels at the ends of the bearing will continue to hold, if the first equalities in them are replaced by

$$\sum_{i=1}^{I} K_i(p_i^e - \langle p\rangle_{ci}) = 0 \ \text{ when } \ K_1B^2 \gg 1 \tag{4.6}$$

$KB^2 \ll 1$ by $K_IB^2 \ll 1$, $KB^2 = O(1)$ by $K_iB^2 = O(1)$ for $i = 1, \ldots, I$ and the terms outside the integral in the last equalities will be replaced by the sum from the left-hand side of Eq. (4.6) multiplied by $B^2/3$.

The analogue of condition (3.9) for an infinite journal bearing for a compressible (for example, isothermal) lubricant is well known [1, 9–11]

$$\int h^3(x)(p_0^2 - p^{\infty 2})dx = 0 \tag{4.7}$$

where $p_0(x)$ is the solution of the corresponding one-dimensional problem.

5. CONCLUSION

Several difficulties arise in attempting a rigorous derivation and justification of the analogues of the conditions obtained above for the case of a compressible barotropic lubricant. The determination of a linear equation for $p_1(x, z)$, which allows of a separation of variables on the basis of (2.1) is only possible by neglecting p_1^2. Similarly, integrals with respect to z from $(\Delta p)^2/B$ have to be neglected in order to obtain the equation for $\langle p\rangle$. Finally, the linear equation for $X_k(x)$ in this case is considerably more complex than condition (2.6), which makes it difficult to prove the non-negativeness of its eigenvalues. If this, nevertheless, is possible, then the required analogues can be easily obtained, since in this case the solutions $Z_k = \operatorname{ch} \mu_k z$ and $Z_k = \operatorname{sh} \mu_k z$ hold. The latter justifies our neglecting the quadratic terms for $B \gg 1$, which are small outside the regions in the immediate vicinity of the ends. The length of such regions $O(1) \ll B$. In extremis the inequality $\lambda_k > 0$ for $k > 0$ can be considered as the hypothesis. It is obvious that even without this factor for $KB^2 \gg 1$ this condition reduces to the first equality of (3.10), and for $KB^2 \ll 1$ it reduces to (4.7) with p_0^2 replaced by $\langle p\rangle^2$. Hence, it only remains to consider the case $KB^2 = O(1)$.

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